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The topological structure of Nieh–Yan form and the chiral anomaly in spaces with torsion

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Abstract. The topological structure of the Nieh–Yan form in a four-dimensional manifold is given by making use of the decomposition of spin connection. The case of the generalized Nieh–Yan form on a 2^d-dimensional manifold is discussed with an example of an eight-dimensional case which is studied in detail. The chiral anomaly with nonvanishing torsion is also studied. The further contributions from the torsional part to the chiral anomaly are found to estimate from the zeros of some fields under the pure gauge condition.

1. Introduction

Torsion might be the most unusual field in physics. Although it has been under investigation for more than two-thirds of a century, there is no general agreement on its mathematical formulation nor on its physical significance. A vast amount of work on torsion has been done by many physicists (see, for example, [1–11]) since it was proposed by E Cartan [12] in the 1920s.

Consider a compact manifold M with metric $g_{\mu\nu}$. There are two dynamically independent 1-forms: the connection $\omega^{ab} = \omega^{ab}_{\mu}\,\mathrm{d}x^{\mu}$ and the vielbein $e^{a} = e^{a}_{\mu}\,\mathrm{d}x^{\mu}$. The curvature and torsion 2-forms are defined by

$$R^{ab} = \mathrm{d}\omega^{ab} - \omega^{ac}\wedge\omega^{cb} \tag{1}$$

$$T^{a} = \mathrm{d}e^{a} - \omega^{ab}\wedge e^{b}. \tag{2}$$

In geometry, ω^{ab} and e^{a} reflect the affine and metric properties of M. While in physics, the curvature and torsion may be related to energy momentum tensor and spin current respectively.

Nieh and Yan [13] first gave the four-dimensional torsional invariant 4-form as

$$N = T^{a}\wedge T^{a} - R^{ab}\wedge e^{a}\wedge e^{b}. \tag{3}$$

This is the only nontrivial locally exact 4-form which vanishes in the absence of torsion and is clearly independent of the Pontryagin and Euler densities. In any local patch where the vielbein is well defined, N can be written as

$$N = \mathrm{d}(e^{a}\wedge T^{a}) \tag{4}$$

and, therefore, is locally exact. The 3-form $e^{a}\wedge T^{a}$ is a Chern–Simons-like form that can be used as a Lagrangian for the dreibein in three dimensions. The dual of this 3-form in four

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dimensions is also known as the totally antisymmetric part of the torsion and is sometimes also referred to as H-torsion,

$$e^a \wedge T^a \wedge \mathrm{d}x^\rho = \varepsilon^{\mu\nu\lambda\rho} T_{\mu\nu\lambda}\, \mathrm{d}^4 x. \tag{5}$$

This component of the torsion tensor is the one that couples to the spin-$\frac{1}{2}$ fields [7].

Recently, there have been some discussions on the question of further contributions to the Chiral anomaly in the presence of space–time with torsion [5–11]. It was shown that the Nieh–Yan form does contribute to the chiral anomaly for a massive field. This additional anomaly term is associated with vacuum polarization diagrams with two external axial torsion vertices, rather than with the usual triangle diagrams [11].

In this paper, by making use of the decomposition theory of $SO(N)$ spin connection (see, for example, [16]) reviewed in section 2, we give the topological structure of the Nieh–Yan form in a four-dimensional manifold in section 3. The Nieh–Yan number is found to be the sum of the indices of some field ϕ at its zeros. The Hopf indices and Brouwer degrees of ϕ label the local properties of the Nieh–Yan form. We also present the relationship between the Nieh–Yan number and the winding number. In section 4, a general discussion on the cases of the generalized Nieh–Yan form on a 2^d-dimensional manifold is given, with an elaborate study of the eight-dimensional case, as an example. By applying the results obtained in this paper and the results obtained by Duan and Fu in [21] to the chiral anomaly, in section 5 it is found that under the pure gauge condition, the contributions from the chiral anomaly only come from the zeros of some field $\tilde{\phi}_{L(R)}$ and ϕ. Finally, we give a short conclusion in section 6. One should note that the main result of this paper is based on reduction to the contribution by singular points, i.e. those where ordinary formulae do not hold.

2. The decomposition theory of spin connection

In this section, we give a short review of the decomposition theory of $SO(N)$ which is a useful tool in the discussion of the topological structure of the Nieh–Yan form used in later sections.

A smooth vector field φ^a $(a = 1, 2, \ldots, N)$ can be found on the base manifold $\boldsymbol{M}$ (a section of a vector bundle over $\boldsymbol{M}$). We define a unit vector n on $\boldsymbol{M}$ as

$$\begin{aligned} &n^a = \varphi^a / \|\varphi\| \qquad a = 1, 2, \ldots, N \\ &\|\varphi\| = \sqrt{\varphi^a \varphi^a} \end{aligned} \tag{6}$$

in which the superscript 'a' is the local orthonormal frame index. In fact, n is identified as a section of the sphere bundle over $\boldsymbol{M}$ (or a partial section of the vector bundle over $\boldsymbol{M}$). We see that the zeros of ϕ are just the singular points of n.

Let the N-dimensional Dirac matrix γ_a $(a = 1, 2, \ldots, N)$ be the basis of the Clifford algebra which satisfies

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}. \tag{7}$$

A unit vector field n on $\boldsymbol{M}$ can be expressed as a vector of Clifford algebra

$$n = n^a \gamma_a. \tag{8}$$

The spin connection 1-form and curvature 2-form are represented as Clifford-algebra-valued differential forms, respectively,

$$\omega = \tfrac{1}{2}\omega^{ab} I_{ab} \qquad F = \tfrac{1}{2} F^{ab} I_{ab} \tag{9}$$

in which I_{ab} is the generator of the spin representation of the group $SO(N)$

$$I_{ab} = \tfrac{1}{4}[\gamma_a, \gamma_b] = \tfrac{1}{4}(\gamma_a \gamma_b - \gamma_b \gamma_a). \tag{10}$$

The covariant derivative 1-form of n^a can be represented in terms of n and ω as

$$\mathrm{D}n = \mathrm{d}n - [\omega, n] \tag{11}$$

and the curvature 2-form as

$$F = \mathrm{d}\omega - \omega \wedge \omega. \tag{12}$$

Arbitrary $U \in Spin(N)$, which satisfies

$$UU^\dagger = U^\dagger U = I \tag{13}$$

is an even versor [14]. The induced 'spinorial' transformation by U to the basis γ_i of the Clifford algebra give N orthonormal vectors u_i [15] via

$$u_i := U\gamma_i U^\dagger = u_i^a \gamma_a \tag{14}$$

where u_i^a is the coefficient of u_i in the representation of Clifford algebra. From the relationship between U and u_i^a, we see that u_i has the same singular points with respect to different 'i'. By (14), it is easy to verify that u_i satisfy

$$u_i u_j + u_j u_i = 2\delta_{ij} \qquad i, j = 1, 2, \ldots N. \tag{15}$$

From (11) we know that the covariant derivative 1-form of u_i is

$$\mathrm{D}u_i = \mathrm{d}u_i - [\omega, u_i]. \tag{16}$$

There exists the following formula for a Clifford algebra r-vector A [14]

$$u_i A u_i = (-1)^r (N - 2r) A. \tag{17}$$

For ω is a Clifford algebra 2-vector and using (17), the spin connection ω can be decomposed by N orthonormal vectors, u_i, as follows:

$$\omega = \tfrac{1}{4}(\mathrm{d}u_i\, u_i - \mathrm{D}u_i\, u_i) \tag{18}$$

or

$$\omega^{ab} = \mathrm{d}u_i^a\, u_i^b - \mathrm{D}u_i^a\, u_i^b. \tag{19}$$

It can be proved that the general decomposition formula (18) has a global property and is independent of the choice of the local coordinates [16].

By choosing the gauge condition

$$\mathrm{D}u_i = 0 \tag{20}$$

we can define a generalized pseudo-flat spin connection as

$$\omega_0 = \tfrac{1}{4}\mathrm{d}u_i\, u_i. \tag{21}$$

Suppose there exist l singular points z_i ($i = 1, 2, \ldots, l.$) in the orthonormal vectors u_j. One can easily prove that at the normal points of u_j

$$F(\omega_0) = 0 \qquad \text{when} \quad x \neq z_i. \tag{22}$$

For the derivative of u_i at the singular points z_i is undefined, formula (22) is invalid at z_i. Hence, the curvature under the gauge condition (20) is a generalized function

$$F \begin{cases} = 0 & \text{when} \quad x \neq z_i \\ \neq 0 & \text{when} \quad x = z_i. \end{cases} \tag{23}$$

This is why we call ω_0 the pseudo-flat spin connection. In fact, the gauge condition (20) is the pure gauge condition supported by the fact that one can always find a frame which is locally flat.

3. Topological structure of Nieh–Yan form in a four-dimensional manifold

One of the properties of topological invariance is that it is independent of connection. Therefore, we choose the pseudo-flat connection

$$\omega^{ab} = \mathrm{d}u_i^a\, u_i^b \tag{24}$$

to make the calculation easier. Under this gauge condition, there must exist singularity points on the manifold if the topology of this manifold is nontrivial. In fact, what we have done is to choose a frame which is locally flat. The choice of frame does not change the topology of the manifold. For the singular property of the pseudo-flat connection, the contribution by the zeros of some field at its singular points will give the topological characteristic. It is a useful method to choose a locally flat frame when dealing with the topological properties of manifolds, see, for example, [22].

Using the pseudo-flat spin connection, we can rewrite the torsion as

$$T^a = \mathrm{D}e^a = \mathrm{D}(e_i u_i^a) = \mathrm{d}e_i\, u_i^a \tag{25}$$

and the Nieh–Yan form

$$N = \mathrm{d}(e^a \wedge T^a) = \mathrm{d}e_i \wedge \mathrm{d}e_i \tag{26}$$

where e_i are the projection of vielbein onto the basis

$$e_i = e^a u_i^a. \tag{27}$$

In quaternionic representation, a 4-vector $\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ is written as

$$\phi = \phi_1 i + \phi_2 j + \phi_3 k + \phi_4 \qquad \phi^* = -\phi_1 i - \phi_2 j - \phi_3 k + \phi_4 \tag{28}$$

and the vielbein projection

$$e = e_1 i + e_2 j + e_3 k + e_4 \tag{29}$$

where $(1, i, j, k)$ is the basis of the quaternion satisfying $i^2 = -1$, $ij = k$, etc. To give a frame that is locally flat, we can find some ϕ, which gives the vielbein expressed in the form

$$e = \frac{l}{\|\phi\|^2}\phi\, \mathrm{d}\phi^* \tag{30}$$

where the constant l has dimensions of length. The choice of a local flat metric is in coincidence with the pseudo-flat connection chosen, which makes singular points exist for the nontrivial topological property of the manifold.

It is well known that the quaternionic representation can be expressed in terms of the Clifford algebra as

$$\phi = \phi_i s^i \qquad i = 1, 2, 3, 4 \tag{31}$$

where

$$s = (\mathrm{i}\vec{\sigma}, I) \qquad s^\dagger = (-\mathrm{i}\vec{\sigma}, I) \tag{32}$$

and

$$e = e_i s^i. \tag{33}$$

The vielbein projection can be rewritten as

$$e = \frac{l}{\|\phi\|^2}\phi\, \mathrm{d}\phi^\dagger. \tag{34}$$

Then we get the Nieh–Yan form as

$$N = \mathrm{d}e_i \wedge \mathrm{d}e_i = \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{d}e \wedge \mathrm{d}e). \tag{35}$$

By making use of the relationship

$$\varepsilon^{ijkl} = \tfrac{1}{2}\,\mathrm{Tr}(s^i s^j s^k s^l) \tag{36}$$

the Nieh–Yan form can be expressed in terms of a unit vector, $n_i = \frac{\phi_i}{\|\phi\|}$, as

$$\begin{aligned} N &= \frac{l^2}{2}\,\mathrm{Tr}(\mathrm{d}n \wedge \mathrm{d}n \wedge \mathrm{d}n \wedge \mathrm{d}n) \\ &= l^2 \varepsilon^{ijkl}\,\mathrm{d}n_i \wedge \mathrm{d}n_j \wedge \mathrm{d}n_k \wedge \mathrm{d}n_l. \end{aligned} \tag{37}$$

The derivative of n_a can be deduced as

$$\mathrm{d}n_i = \frac{\mathrm{d}\phi_i}{\|\phi\|} - \phi_i\,\mathrm{d}\left(\frac{1}{\|\phi\|}\right). \tag{38}$$

Substituting it into (37), we obtain the expression of N on $S(\boldsymbol{M})$

$$N = l^2 \varepsilon^{ijkl}\,\mathrm{d}\left(\frac{\phi_i}{\|\phi\|^4}\,\mathrm{d}\phi_j \wedge \mathrm{d}\phi_k \wedge \mathrm{d}\phi_l\right). \tag{39}$$

Using

$$\frac{\phi_i}{\|\phi\|^4} = -\frac{1}{2}\frac{\partial}{\partial\phi_i}\left(\frac{1}{\|\phi\|^2}\right) \tag{40}$$

equation (39) becomes

$$N = -\frac{l^2}{2}\varepsilon^{ijkl}\,\frac{\partial}{\partial\phi_m}\frac{\partial}{\partial\phi_i}\left(\frac{1}{\|\phi\|^2}\right)\frac{\partial\phi_m}{\partial x^\mu}\frac{\partial\phi_j}{\partial x^\nu}\frac{\partial\phi_k}{\partial x^\lambda}\frac{\partial\phi_l}{\partial x^\rho}\frac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{g}}\sqrt{g}\,\mathrm{d}^4x \tag{41}$$

where $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ is the metric tensor of $\boldsymbol{M}$. Defining the Jacobian $\mathrm{D}(\phi/x)$ as

$$\varepsilon^{ijkl}\,\mathrm{D}(\phi/x) = \epsilon^{\mu\nu\lambda\rho}\frac{\partial\phi_i}{\partial x^\mu}\frac{\partial\phi_j}{\partial x^\nu}\frac{\partial\phi_k}{\partial x^\lambda}\frac{\partial\phi_l}{\partial x^\rho} \tag{42}$$

and noting that

$$\varepsilon^{ijkl}\varepsilon^{mjkl} = 6\delta^{im} \tag{43}$$

we get

$$N = -3l^2\frac{\partial^2}{\partial\phi_i\partial\phi_i}\left(\frac{1}{\|\phi\|^2}\right)\mathrm{D}\left(\frac{\phi}{x}\right)\mathrm{d}^4x. \tag{44}$$

The general Green function formula [17] in ϕ space is

$$\frac{\partial^2}{\partial\phi_i\partial\phi_i}\left(\frac{1}{\|\phi\|^2}\right) = -4\pi^2\delta(\phi). \tag{45}$$

We obtain the new formulation of the Nieh–Yan form in terms of the δ function, $\delta(\phi)$:

$$N = 12\pi^2 l^2\delta(\phi)\,\mathrm{D}(\phi/x)\,\mathrm{d}^4x. \tag{46}$$

Suppose $\phi(x)$ has l isolated zeros on $\boldsymbol{M}$ and let the ith zero be z_i, it is well known from the ordinary δ-function theory [18] that

$$\delta(\phi) = \sum_{i=1}^{l}\frac{\beta_i\delta(x-z_i)}{\mathrm{D}(\phi/x)|_{x=z_i}}. \tag{47}$$

Then one obtains

$$\delta(\phi)\,\mathrm{D}\left(\frac{\phi}{x}\right) = \sum_{i=1}^{l}\beta_i\eta_i\delta(x-z_i) \tag{48}$$

where β_i is the positive integer (the Hopf index of the ith zero) and η_i the Brouwer degree [19, 20]:

$$\eta_i = \operatorname{sgn} \mathrm{D}(\phi/x)|_{x=z_i} = \pm 1. \tag{49}$$

From the above deduction the following topological structure is obtained:

$$n_{N-Y} = 12\pi^2 l^2 \delta(\phi)\, \mathrm{D}\left(\frac{\phi}{x}\right) \mathrm{d}^4 x = 12\pi^2 l^2 \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i)\, \mathrm{d}^4 x \tag{50}$$

which means that the local structure of N is labelled by the Brouwer degrees and Hopf indices, which are topological invariants. Therefore, the Nieh–Yan number n_{N-Y} can be represented as

$$n_{N-Y} = \int_{M} N = 12\pi^2 l^2 \sum_{i=1}^{l} \beta_i \eta_i. \tag{51}$$

On the other hand, we can decompose $\boldsymbol{M}$ as

$$\boldsymbol{M} = \sum_i \boldsymbol{M}_i \tag{52}$$

so that $\boldsymbol{M}_i$ includes only the ith singularity point z_i of $n(x)$. Then we get

$$\begin{aligned} n_{N-Y} &= \sum_i \int_{\boldsymbol{M}_i} l^2 \varepsilon^{ijkl}\, \mathrm{d}n_i \wedge \mathrm{d}n_j \wedge \mathrm{d}n_k \wedge \mathrm{d}n_l \\ &= \sum_i \oint_{\partial \boldsymbol{M}_i} l^2 \varepsilon^{ijkl} n_i \wedge \mathrm{d}n_j \wedge \mathrm{d}n_k \wedge \mathrm{d}n_l \end{aligned} \tag{53}$$

where $\partial \boldsymbol{M}_i$ is the boundary of $\boldsymbol{M}_i$. Equation (53) is another definition of the winding number $W(\phi, z_i)$ of the surface $\partial \boldsymbol{M}_i$ and the mapping $\phi(x)$ [23]

$$W(\phi, z_i) = \frac{1}{12\pi^2} \oint_{\partial \boldsymbol{M}_i} \varepsilon^{ijkl} n_i \wedge \mathrm{d}n_j \wedge \mathrm{d}n_k \wedge \mathrm{d}n_l = \beta_i \eta_i. \tag{54}$$

Then, the Nieh–Yan number n_{N-Y} can be expressed further in terms of the winding numbers

$$n_{N-Y} = 12\pi^2 l^2 \sum_{i=1}^{l} W(\phi, z_i). \tag{55}$$

The sum of the winding numbers can be interpreted or, indeed, defined as the degree of the mapping $\phi(x)$ onto $\boldsymbol{M}$. By (46) and (53), we have

$$\begin{aligned} \sum_{i=1}^{l} W(\phi, z_i) &= \int_{M} \delta(\phi)\, \mathrm{D}\left(\frac{\phi}{x}\right) \mathrm{d}^4 x \\ &= \deg \phi \int_{\phi(M)} \delta(\phi)\, \mathrm{d}^4 \phi \\ &= \deg \phi. \end{aligned} \tag{56}$$

Therefore, we finally get the Nieh–Yan number

$$n_{N-Y} = 12\pi^2 l^2 \sum_{i=1}^{l} W(\phi, z_i) = 12\pi^2 l^2 \deg \phi. \tag{57}$$

From (6), we know that the zeros of ϕ are just the singularities of n. Here, (51) implies that the sum of the indices of the singular points of n, or of the zeros of ϕ, is the Nieh–Yan number. One should notice that the vector field ϕ is not a section of the tangent bundle of $\boldsymbol{M}$, i.e. ϕ is not a Riemannian vector field. Therefore, formula (57) is not concerned with the Poincaré–Hopf theorem which connects the Euler characteristics with the zeros of a Riemannian vector field.

4. Higher-dimensional case

In a 2^d-dimensional case, we can get similar results. Under the pure gauge condition, the nonvanishing term of the generalized Nieh–Yan form is

$$(\mathrm{d}e_i \wedge \mathrm{d}e_i)^{2^{d-2}}. \tag{58}$$

The computation above is also valid in this case. The topological structure of generalized Nieh–Yan form is constituted by the δ function of some ϕ. The Hopf indices and Brouwer degree label the local structure of the Nieh–Yan form. The Nieh–Yan number is some constant number times the degree or winding numbers of ϕ:

$$N \sim \delta(\phi)\, \mathrm{D}\left(\frac{\phi}{x}\right) \mathrm{d}^{2^d} x = \sum_i \beta_i \eta_i \delta(x - z_i)\, \mathrm{d}^{2^d} x. \tag{59}$$

In the following, we only take the eight-dimensional case as an example.

In an eight-dimensional compact manifold, under gauge condition (20), the generalized Nieh–Yan form is expressed simply as

$$N = \mathrm{d}(e_i \wedge \mathrm{d}e_i \wedge \mathrm{d}e_j \wedge \mathrm{d}e_j). \tag{60}$$

And the vielbein is expressed locally in terms of the octonions under Clifford algebra representation

$$e = e_i s^i = \frac{l}{\|\phi\|^2} \phi\, \mathrm{d}\phi^\dagger \tag{61}$$

where

$$\phi = \phi_i s^i \qquad i = 1, 2, \ldots, 8$$

in which s^i is the basis of octonions. Substituting (61) into (60), we get the Nieh–Yan form in eight dimensions in terms of the unit vector $n_i = \frac{\phi_i}{\|\phi\|}$ as

$$N = \frac{l^4}{5} \varepsilon^{i_1 i_2 \ldots i_8}\, \mathrm{d}n_{i_1} \wedge \mathrm{d}n_{i_2} \wedge \ldots \wedge \mathrm{d}n_{i_8}. \tag{62}$$

Analogously to the four-dimensional case, using the general Green function formula [17] in eight-dimensional ϕ space, the generalized Nieh–Yan form is expressed as

$$\begin{aligned} N &= 1008 l^4 A(S^7) \delta(\phi)\, \mathrm{D}\left(\frac{\phi}{x}\right) \mathrm{d}^8 x \\ &= 336\pi^4 l^4 \delta(\phi) \mathrm{D}\left(\frac{\phi}{x}\right) \mathrm{d}^8 x \end{aligned} \tag{63}$$

which can be rewritten in terms of Hopf indices and Brouwer degree as

$$N = 336\pi^4 l^4 \sum_i \beta_i \eta_i \delta(x - z_i)\, \mathrm{d}^8 x. \tag{64}$$

The Nieh–Yan number in the eight-dimensional case is

$$n_{N-Y} = 336\pi^4 l^4 \deg \phi = 336\pi^4 l^4 \sum_i W(\phi, z_i). \tag{65}$$

Similar results can be obtained for higher 2^d-dimensional cases.

5. Chiral anomaly on spaces with torsion

Consider a massive Dirac spinor on a curved background with torsion. The action is

$$S = \int \tfrac{1}{2}(\mathrm{d}^4x\, e\bar{\psi}\,\nabla\!\!\!\!/\,\psi + \text{h.c.}) + m\bar{\psi}\psi \tag{66}$$

where the Dirac operator is given by

$$\nabla\!\!\!\!/ = e^{\mu a}\gamma_a D_\mu. \tag{67}$$

This action is invariant under rigid chiral transformations

$$\psi' \to \mathrm{e}^{\mathrm{i}\varepsilon\gamma_5}\psi \tag{68}$$

where ε is a real constant parameter. This symmetry leads to the classical conservation law

$$\partial_\mu J_5^\mu = 0 \tag{69}$$

in which

$$J_5^\mu = e e^{\mu a}\bar{\psi}\gamma_a\gamma_5\psi. \tag{70}$$

The chiral anomaly when torsion is present is given by [5, 6, 8, 11]

$$\partial_\mu\langle J_5^\mu\rangle = A(x) \tag{71}$$

with

$$A(x) = \frac{1}{8\pi^2} * \left[R^{ab}\wedge R^{ab} + \frac{2}{l^2}(T^a\wedge T^a - R^{ab}\wedge e^a\wedge e^b)\right]. \tag{72}$$

The constant l is called the radius of the universe and is related to the cosmological constant ($|\Lambda| = l^{-2}$).

Using the results obtained in section 3, we can rewrite the chiral anomaly as

$$\partial_\mu\langle J_5^\mu\rangle = \frac{1}{8\pi^2} * [R^{ab}\wedge R^{ab}] + 3\delta(\phi)\,\mathrm{D}\left(\frac{\phi}{x}\right). \tag{73}$$

From the relationship

$$so(4) = su(2)_L \times su(2)_R \tag{74}$$

the Pontryagin class of the $SO(4)$ group can be expressed as the sum of the second Chern classes of the left and right $SU(2)_{L(R)}$ subgroup

$$\mathrm{Tr}(R\wedge R) = \mathrm{Tr}(R_{SU(2)_L}\wedge R_{SU(2)_L}) + \mathrm{Tr}(R_{SU(2)_R}\wedge R_{SU(2)_R}). \tag{75}$$

By making use of the result of Duan and Fu [21]:

$$\frac{1}{8\pi^2}\,\mathrm{Tr}(R_{SU(2)}\wedge R_{SU(2)}) = \delta(\tilde{\phi})\,\mathrm{D}\left(\frac{\tilde{\phi}}{x}\right)\,\mathrm{d}^4x \tag{76}$$

where $\tilde{\phi}$ belongs to $Spin(3)$ corresponding to the group $SU(2)$. In Duan and Fu's paper, the pure gauge condition is also used to get the topological structure of the second Chern class for $SU(2)$ group. We now get the chiral anomaly

$$\partial_\mu\langle J_5^\mu\rangle = \delta(\tilde{\phi}_L)\,\mathrm{D}\left(\frac{\tilde{\phi}_L}{x}\right) + \delta(\tilde{\phi}_R)\,\mathrm{D}\left(\frac{\tilde{\phi}_R}{x}\right) + 3\delta(\phi)\,\mathrm{D}\left(\frac{\phi}{x}\right) \tag{77}$$

in which $\tilde{\phi}_L$ and $\tilde{\phi}_R$ are some $Spin(3)_{L(R)}$ elements corresponding to the subgroups $SU(2)_{L(R)}$. Furthermore, by making use of the structure of delta function, the anomaly can be formulated more explicitly:

$$\begin{aligned}\partial_\mu \langle J_5^\mu \rangle &= \sum_i \delta(x - z_{Li})\beta_{Li}\eta_{Li} + \sum_i \delta(x - z_{Ri})\beta_{Ri}\eta_{Ri} + 3\sum_i \delta(x - z_i)\beta_i\eta_i \\ &= \sum_i \delta(x - z_{Li})W_{Li} + \sum_i \delta(x - z_{Ri})W_{Ri} + 3\sum_i \delta(x - z_i)W_i \qquad (78)\end{aligned}$$

where $\beta_{L(R)i}$ and $\eta_{L(R)i}$ are the Hopf indices and Brouwer degrees, respectively, corresponding to the zeros of $\tilde{\phi}_{L(R)}$, and $W_{L(R)i}$ is the winding number of $\tilde{\phi}_{L(R)}$ at its ith zeros. From (78), we see, under the pure gauge condition, that the chiral anomaly comes only from the zeros of the fields $\tilde{\phi}_{LR}$ and ϕ, and that their winding numbers account for the magnitude of the chiral anomaly.

6. Conclusion

In this paper, we have discussed the Nieh–Yan form by making use of the decomposition theory of spin connection. The Nieh–Yan form in a four-dimensional manifold is proved to take δ-function form under a locally flat gauge condition. The local topological structure of the Nieh–Yan form is labelled by Hopf indices and Brouwer degrees of field ϕ, which is used to express the vielbein under a locally flat (or pure) gauge condition. The Nieh–Yan number is proved to be a constant number times the degree or winding number of ϕ.

A general discussion of the generalized Nieh–Yan form on a 2^d-dimensional manifold is presented. From the example of an eight-dimensional case, we find the topological structure of the generalized Nieh–Yan form is similar to the four-dimensional case in terms of the field ϕ under the pure gauge condition. It is noticeable that the Clifford algebra can make the calculus simpler in these cases.

The topological anomaly with nonvanishing torsion is formulated explicitly under the pure gauge condition. There are two kinds of contribution in the anomaly which comes from the topology of the manifold: the Pontryagin class of $SO(4)$ and the Nieh–Yan form. We have proved each of them to be a sum of δ functions of some fields, $\tilde{\phi}_{L(R)}$ and ϕ, under the pure gauge condition. This means the contribution comes only from the zeros of $\tilde{\phi}_{L(R)}$ and ϕ. The degrees or winding numbers of $\tilde{\phi}_{L(R)}$ or ϕ give the magnitude of the contributions.

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